Exploring bicentric polygons

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Abstract

A bicentric polygon *is a polygon which is simultaneously* cyclic*: all vertices lie on a single circle, and* tangential*: all edges are tangential to another circle. All triangles are bicentric, as are all regular polygons. However, non-regular bicentric polygons of orders four or higher have some interesting properties, and they have been the subject of study and interest since the time of Euler. In particular the radii of the two circles, and of the distance between their centres, satisfy formulas which get progressively more complex as the number of sides increases. In this article we show how such formulas—and some fine diagrams—can be obtained with dynamic geometry software. This brings a somewhat niche area of geometry into the realm of modern experimentation, and what has previously required some very complicated algebra into the reach of an able school student.*

1 Introduction

In 1767, the Swiss mathematician Leonhard Euler proved what is now called *Euler's triangle theorem*; that for any triangle the radii r and R of the incircle and circumcircle respectively, and of the distance d between their centres, were related by the formula

$$R(R-2r)=d^2.$$

Sometime later, Euler's assistant Nicholas Fuss showed that for a bicentric quadrilateral, the relation is

$$2r^2(R^2+d^2)=(R^2-d^2)^2.$$

These formulas can be written also as

$$\frac{1}{R+d}+\frac{1}{R-d}=\frac{1}{r} \qquad \text{and} \qquad \frac{1}{(R+d)^2}+\frac{1}{(R-d)^2}=\frac{1}{r^2}. \tag{1}$$

It might be hoped that the corresponding formula for a bicentric pentagon would follow a similar pattern, but as we shall see the result is more complex.

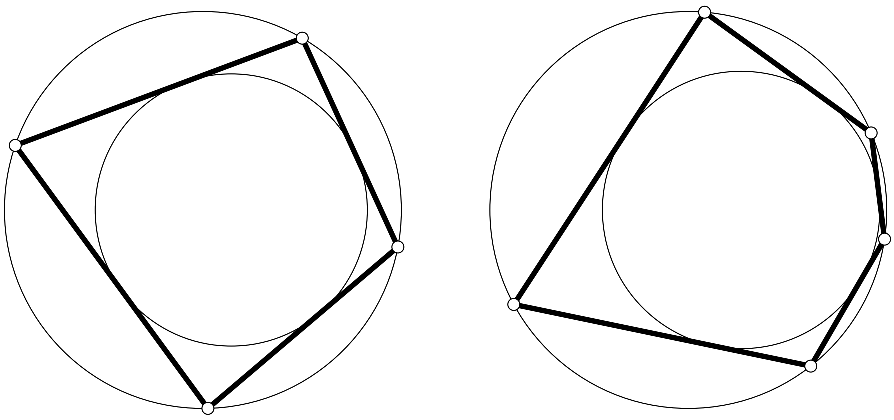
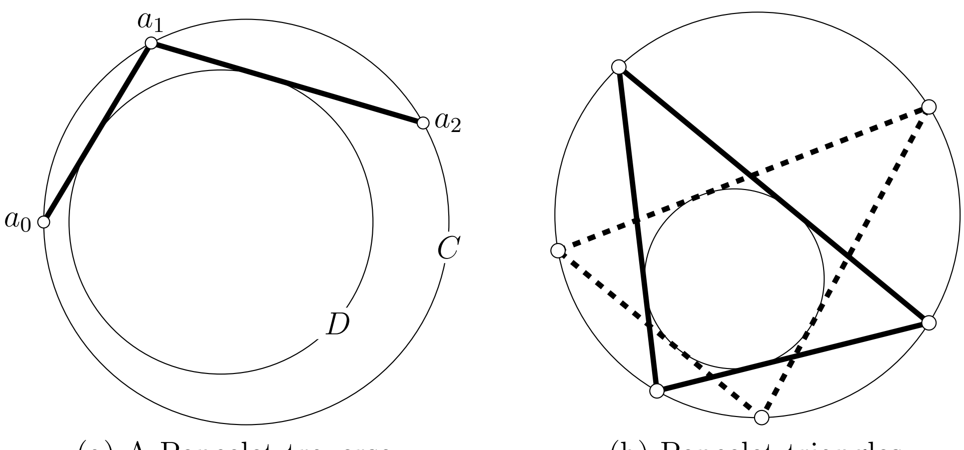


Figure 1: Bicentric polygons

Examples of a bicentric quadrilateral and pentagon are shown in figure 1.

One of the most remarkable results about bicentric polygons is named for the French mathematician and engineer Jean-Victor Poncelet (1788–1867), and is called *Poncelet's Porism*, or *Poncelet's Closure Theorem*. A "porism" is a result which if true, is true for an infinite number of states. In Poncelet's case, we start by defining a *Poncelet traverse*; this starts with two circles C and D, with D inside C, and in general not concentric. In fact the result is true for any conic sections, not just circles. However, for ease of exposition we shall restrict our attention to circles. Pick any point a_0 on the outer circle C, and from that point draw a tangent to the inner circle D to meet the outer circle C in another point a_1. Then from a_1 draw a tangent to D to meet C at a_2, and so on. Poncelet's result is that if this path if closed in the sense that $a_n = a_0$ for some value n, then such a path will be closed for *all* starting points on the outer circle.

Figure 2(a) shows the beginning of such a traverse.



(a) A Poncelet traverse (b) Poncelet triangles

Figure 2: Poncelet's porism

Poncelet's porism is a highly non-trivial result; however a very accessible proof has been given byUeno, Shiga, and Morita [8]. More recently, an elementary combinatorial proof has been given by Halbeisen and Hungerbhler [3].

One immediate corollary is that if D and C are the incircle and circumcircle respectively of a particular triangle, then they will be the incircle and circumcircle of an infinite number of other triangles. Figure 2(b) demonstrates this.

2 Using a computer system

To explore bicentric polygons, and the relations between the circles, any reasonably full-featured computer system can be used, as long as it supports both geometry, and some computer algebra (CAS). For example we could use Giac/Xcas [7], or any of the recent versions of GeoGebra [5] which include a CAS.

For exposition we shall choose GeoGebra; being open-source it's readily accessible, and the CAS included is Giac (of which, however, only a limited number of commands are available through the GeoGebra front end). Note that some elegant GeoGebra animations have been made available online by Borcherds [1].

Following Kerawala [6], we start by developing the relationship between the ends of a tangent (such as in a Poncelet traverse), and the radii of the circles and the distance between their centres. We shall position the outer circle so that its centre is on the origin, and the inner circle to have centre at $(d, 0)$. Then the general position of a tangent is shown in figure 3.

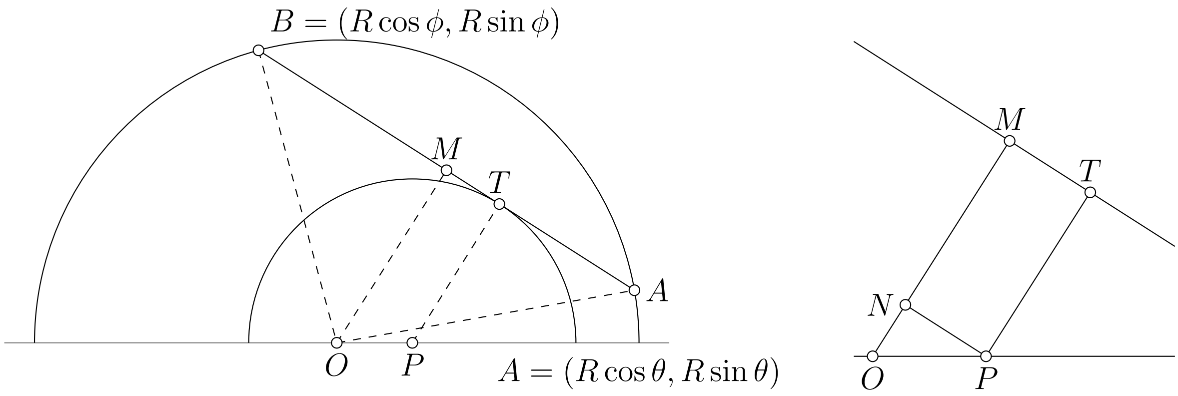


Figure 3: Determining a relation between angles on a tangential chord

In these figures AB is the tangent, and T is the point of tangency to the inner circle. Since angle $BOA = \phi - \theta$ it follows that angle $BOM = (\phi - \theta)/2$ and so angle $MOP = (\phi + \theta)/2$. Considering the line OM in the second figure, we have $ON + NM = OM$ and expanding these in terms of the angles involved we obtain the expression

$$d\cos\left(\frac{\phi+\theta}{2}\right) + r = R\cos\left(\frac{\phi+\theta}{2}\right),$$

or

$$d\cos\left(\frac{\phi}{2}+\frac{\theta}{2}\right) + r = R\cos\left(\frac{\phi}{2}+\frac{\theta}{2}\right). \tag{2}$$

This last expression can be turned into a polynomial by using the well known tangent half-angle substitution for $\theta/2$ and $\phi/2$:

$$s = \tan\left(\frac{\theta/2}{2}\right), \qquad t = \tan\left(\frac{\phi/2}{2}\right)$$

Substituting $\theta = 4\tan^{-1}(s)$ and $\phi = 4\tan^{-1}(t)$ into equation 2 produces (after some elementary algebra) the expression

$$4(R+d)st + (R-d)(1-s^2)(1-t^2) = r(1+s^2)(1+t^2). \tag{3}$$

This equation has been given in a slightly different form by Kerawala [6], and has been called the "quadratic involution".

Note that this algebra can in fact be done using the CAS View of GeoGebra 5.0:

ex1:= cos(2*atan(s)-2*atan(t)) $\rightarrow \quad \mathbf{ex1} := \mathbf{cos}\,(\mathbf{2}\ \mathbf{atan}\,(\mathbf{s}) - \mathbf{2}\ \mathbf{atan}\,(\mathbf{t}))$
Simplify[TrigExpand[ex1]] $\rightarrow \quad \dfrac{\mathbf{s^2\ t^2 - s^2 + 4\ s\ t - t^2 + 1}}{\mathbf{s^2\ t^2 + s^2 + t^2 + 1}}$
ex2:= cos(2*atan(s)+2*atan(t)) $\rightarrow \quad \mathbf{ex2} := \mathbf{cos}\,(\mathbf{2}\ \mathbf{atan}\,(\mathbf{s}) + \mathbf{2}\ \mathbf{atan}\,(\mathbf{t}))$
Simplify[TrigExpand[ex2]] $\rightarrow \quad \dfrac{\mathbf{s^2\ t^2 - s^2 - 4\ s\ t - t^2 + 1}}{\mathbf{s^2\ t^2 + s^2 + t^2 + 1}}$

From the above we can write

$$\cos\left(2\tan^{-1}(s) - 2\tan^{-1}(t)\right) = \frac{(s^2-1)(t^2-1)+4st}{(s^2+1)(t^2+1)},$$
$$\cos\left(2\tan^{-1}(s) + 2\tan^{-1}(t)\right) = \frac{(s^2-1)(t^2-1)-4st}{(s^2+1)(t^2+1)}.$$

This means that equation (2) can be rewritten as

$$d\frac{(s^2-1)(t^2-1)-4st}{(s^2+1)(t^2+1)} + r = R\frac{(s^2-1)(t^2-1)+4st}{(s^2+1)(t^2+1)}.$$

Multiplying through by $(s^2+1)(t^2+1)$ and re-arranging will produce the quadratic involution of equation (3).

The involution equation is at the heart of all work with bicentric polygons, and it can be written in several different forms. From equation (1) we could make the substitutions

$$a = \frac{1}{R+d}, \quad b = \frac{1}{R-d}, \quad c = \frac{1}{r}$$

with which Euler's and Fuss's formulas have the very simple forms

$$a + b = c, \quad a^2 + b^2 = c^2$$

respectively. Since we are dealing with planar figures, we can scale a bicentric polygon and its circles so that $r = 1$. In effect this means multiplying Euler's formula by r and Fuss's formula by r^2 to obtain:

$$\frac{r}{R-d} + \frac{r}{R+d} = 1, \quad \frac{r^2}{(R-d)^2} + \frac{r^2}{(R+d)^2} = 1.$$

Making the substitutions

$$p = \frac{r}{R+d}, \quad q = \frac{r}{R-d}$$

gives the very simple forms

$$p + q = 1, \quad p^2 + q^2 = 1.$$

By substitution we find that the "*abc*" form of the involution is

$$4bcst + ac(1-s^2)(1-t^2) = ab(1+s^2)(1+t^2) \tag{4}$$

and the "*pq*" form of the involution is

$$4qst + p(1-s^2)(1-t^2) = pq(1+s^2)(1+t^2). \tag{5}$$

Note that our definition of p and q is non-standard; an earlier definition is

$$p = \frac{R-d}{r}, \quad q = \frac{R+d}{r}$$

and this is given by Weisstein [9] and credited to the 19th century German mathematician Friedrich Julius Richelot. However, our definition has the advantage of producing extremely simple and elegant results.

3 Euler's and Fuss's formulas

By Poncelet's porism, we can start a traverse (for a triangle or quadrilateral) at the point $A = (R, 0)$. For a triangle this will mean that the opposite side is vertical; for a quadrilateral this means there will be an opposite point at $(-R, 0)$, as shown in figure 4.

For the triangle, the right most point will have $s = 0$, for the other two points will have values s and $-s$. Writing the involution as

$$T(s,t,r,R,d) = 4(R+d)st + (R-d)(1-s^2)(1-t^2) - r(1+s^2)(1+t^2),$$

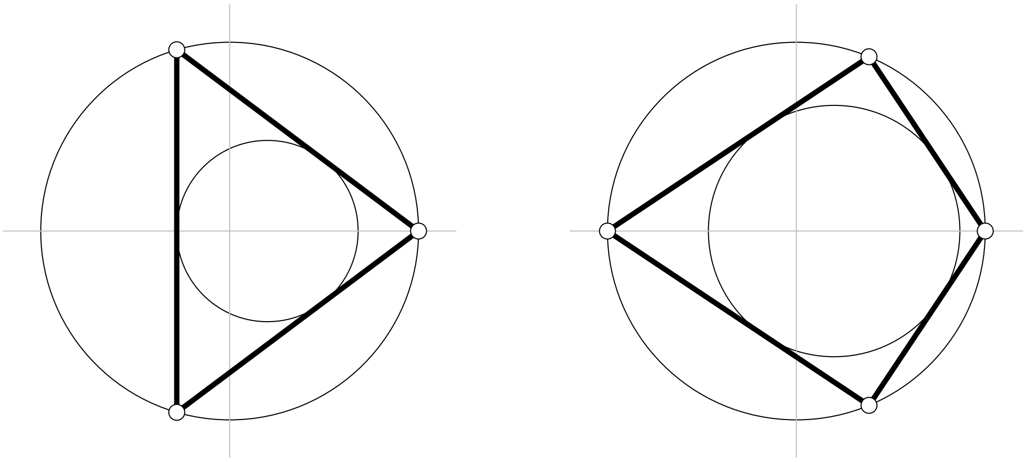


Figure 4: Setting up to obtain Euler's and Fuss's formulas

so that being a tangent means that $T(s, t, r, R, d) = 0$, we can define the triangle as

$$T(0, t, r, R, d) = 0,$$
$$T(t, -t, r, R, d) = 0.$$

Eliminating t from these equations should produce Euler's formula. Using a recent GeoGebra (version 5.0 or above), the CAS view includes all the necessary mathematical tools:

tngt(s,t,r,R,d) := 4*(R+d)*s*t+(R-d)*(1-s^2)*(1-t^2)-r*(1+s^2)*(1+t^2)
e1:=Eliminate[{tngt(0, t,r,R,d),tngt(t,-t,r,R,d)},{t}] $\rightarrow \quad \{\mathbf{R^3 - R^2d - Rd^2 + d^3 + R^2r - 2Rdr + d^2r - 2Rr^2}\}$
Factors[Element[e1,1]] $\rightarrow \quad \begin{pmatrix} \mathbf{r+d-R} & \mathbf{1} \\ \mathbf{2rR - d^2 + R^2} & \mathbf{1} \\ \mathbf{-1} & \mathbf{1} \end{pmatrix}$

This means that the elimination of t produces the expression

$$(r + d - R)(2rR - d^2 + R^2)(-1) = 0.$$

The first term can be rewritten $r + d = R$; this corresponds to the circles meeting at the point $(R, 0)$, and the resulting triangle is "degenerate" in that all sides have zero length and all vertices are at $(R, 0)$. The second term is the non-degenerate case, and the expression can be written as $2rR + R^2 = d^2$ which is Euler's formula.

To obtain Fuss's formula, note that the point at $(-R, 0)$ has $t = 1$, so that the expressions will be

$$T(0, t, r, R, d) = 0,$$
$$T(t, 1, r, R, d) = 0.$$

Again using the CAS view:

e2:=Eliminate[{tngt(0, t,r,R,d),tngt(t,1,r,R,d)},{t}] $\rightarrow \quad \{\mathbf{R^4 - 2R^2d^2 + d^4 - 2R^2r^2 - 2d^2r^2}\}$

This expression can't be factorized, but:

ex1:=Element[e2,1]
Factors[ex1-(R^2-d^2)^2] $\rightarrow \quad \begin{pmatrix} \mathbf{2} & \mathbf{1} \\ \mathbf{r} & \mathbf{2} \\ \mathbf{R^2+d^2} & \mathbf{1} \\ \mathbf{-1} & \mathbf{1} \end{pmatrix}$

This means that the formula for a bicentric quadrilateral can be written

$$(R^2 - d^2)^2 = 2r^2(R^2 + d^2)$$

which is Fuss's formula.

Drawing cyclic quadrilaterals

In GeoGebra, a bicentric quadrilateral can be drawn by first drawing the two circles to satisfy Fuss's formula. Then by Poncelet's porism, choose any point on the outer circle, and construct a traverse by alternate use of the tangent and intersection tools.

To define the circles, pick a triplet (x, y, z) for which $x^2+y^2 = z^2$, for example $5^2+12^2 = 13^2$, although the triplet does not have to consist of integers. Using the form given in equation (1), we have the equations

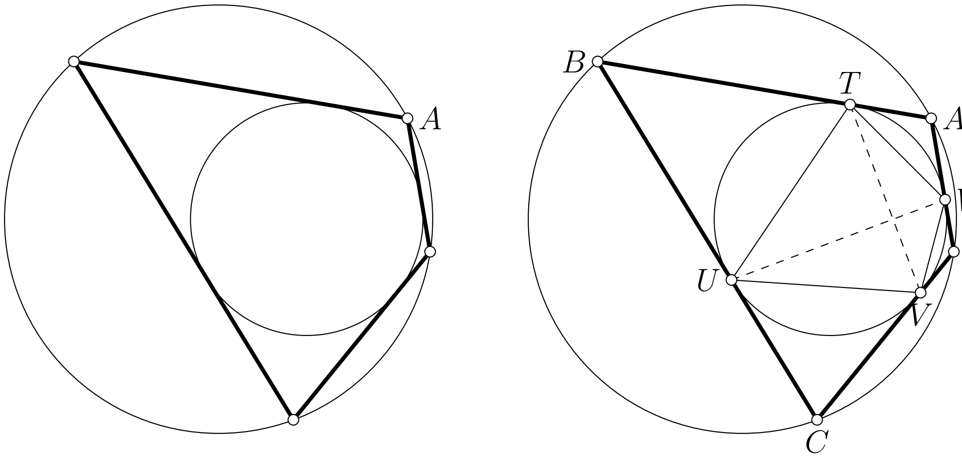
$$R + d = \frac{1}{5}, \quad R - d = \frac{1}{12}, \quad r = \frac{1}{13}.$$

The first two equations can be solved to obtain $R = 17/120$, $d = 7/120$. The two circles are then

$$x^2 + y^2 = \left(\frac{17}{120}\right)^2, \quad \left(x - \frac{7}{120}\right)^2 + y^2 = \left(\frac{1}{13}\right)^2.$$

Starting at the point $A = (15/120, 8/120) = (1/8, 1/15)$ on the outer circle produces the quadrilateral shown in figure 5(a).

Another method of creating a bicentric quadrilateral starts by considering the points of tangency on the inner circle. First note that in a cyclic quadrilateral, the opposite angles add to π; this is an elementary result found in Euclid (Book III, Prop. 22); see for example the translation by Heath [4]. Consider the inner circle and its points of tangency as shown in figure 5(b).



(a) Starting with two circles (b) The inner chords

Figure 5: Constructing cyclic quadrilaterals

Suppose that the angles at each of the quadrilateral vertices are A, B, C and D. Then the lengths of the chords WT, TU, UV and VW are

$$2r\cos\left(\frac{A}{2}\right),\quad 2r\cos\left(\frac{B}{2}\right),\quad 2r\cos\left(\frac{C}{2}\right),\quad 2r\cos\left(\frac{D}{2}\right)$$

respectively. Then

$$\begin{aligned}
TW^2+UV^2 &= 4r^2\left(\cos^2\left(\frac{A}{2}\right)+\cos^2\left(\frac{C}{2}\right)\right)\\
&= 4r^2\left(\cos^2\left(\frac{A}{2}\right)+\sin^2\left(\frac{A}{2}\right)\right)\\
&\qquad \text{since } \cos(C/2)=\cos(\pi/2-A/2)=\sin(A/2)\\
&= 4r^2.
\end{aligned}$$

Similarly $TU^2+VW^2=4r^2$, and so $TW^2+UV^2=TU^2+VW^2$. It is a standard result that if the sums of squares of opposite sides of a quadrilateral are equal, then the diagonals are perpendicular. Hence in a bicentric quadrilateral, the chords between opposite tangent points are perpendicular.

This means that another method of obtaining a bicentric quadrilateral is to start with two perpendicular chords in a circle. Extend the tangents from each end of each chord until they meet; those points will be the vertices of a bicentric quadrilateral. This is demonstrated in figure 6.

4 Bicentric pentagons

Just as for Euler's and Fuss's formulas, the relationship between the radii and distances can be determined by using the involution function. However, to keep the algebra manageable, we

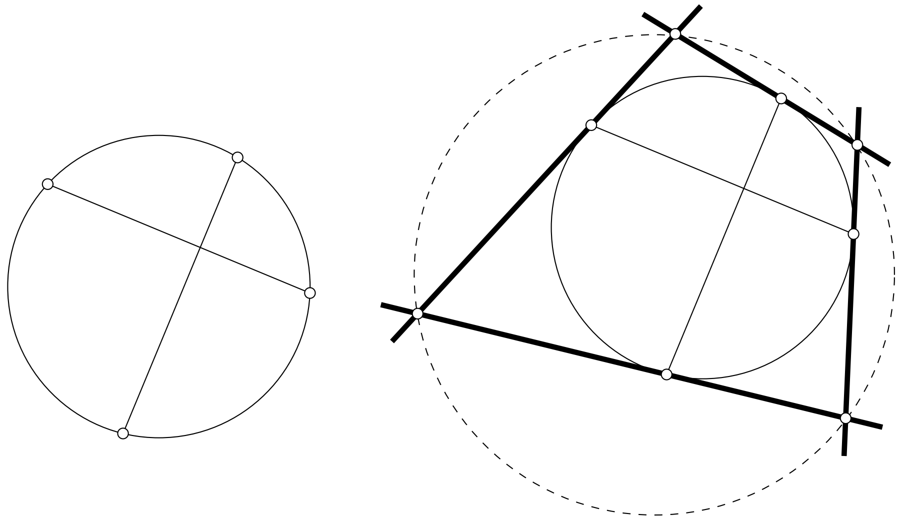


Figure 6: Creating a bicentric quadrilateral starting with perpendicular chords

shall use the pq form given in equation (5). Starting with $A = (R, 0)$ and with $s = 0$, there will be a second point before the vertical tangent. Thus:

$$\begin{aligned} T(0,t,p,q) &= 0, \\ T(t,u,p,q) &= 0, \\ T(u,-u,p,q) &= 0. \end{aligned}$$

Eliminating t and u from these equations produces the expression

$$p(p-1)^2q(q+p+1)(q^3-pq^2-q^2-qp^2-2pq-q+p^3-p^2-p+1)=0$$

of which the last term is the one we want. The other terms correspond to degenerate pentagons, when one or more edges have length of zero, so that their endpoints coincide. This term can be written as

$$p^3+q^3-(p+q)(p+1)(q+1)+1=0 \implies p^3+q^3+1=(p+q)(p+1)(q+1).$$

In order to plot a bicentric pentagon, start by plotting this curve, which can be performed in GeoGebra by using the ImplicitCurve command, in which p and q are replaced with x and y:

```
c:=ImplicitCurve[x\^{}3+y\^{}3-(x+y)(x+1)(y+1)+1]}
```

and which is shown in figure 7.

To construct a bicentric polygon, we choose any point (x, y) on the curve, and assuming $r = 1$, find R and d from the substitution equations, from which

$$R=\frac{x+y}{2xy}, \quad d=\frac{x-y}{2xy}$$

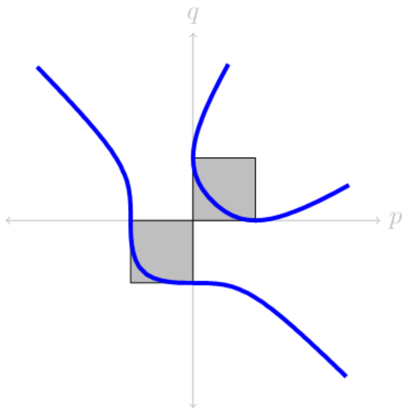


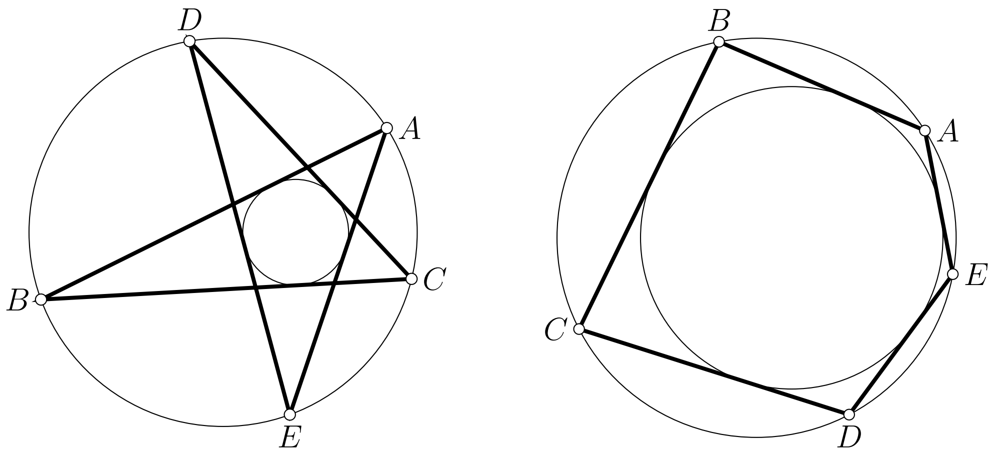
Figure 7. The p, q relation for circles to admit a bicentric pentagon

R:=(x(A)+y(A))/(2*x(A)*y(A))
d:=(x(A)-y(A))/(2*x(A)*y(A))

Then in the Algebra View we can enter

```
c1:=Circle[(0,0),R]
c2:=Circle[(d,0),1]
```

Then pick any point on the outer circle, and construct a Poncelet traverse. For example, if $(x, y) = (0.4354, 0.1985)$ then $R = 3.6679$ and $d = 1.3711$. Starting with $A = (3.0937, 1.9705)$ the resulting pentagon is shown in figure 7(a). This is a remarkable result: a non-convex bi-



(a) $(p, q) = (0.4354, 0.1985)$ (b) $(p, q) = (-0.6442, -0.9195)$

Figure 7: Two bicentric pentagons

centric polygon; in fact a bicentric pentagram. However, it satisfies Poncelet's porism perfectly well: if the point A is dragged around the circle, the pentagram obligingly follows. (Note that a five-pointed star is omitted from the animations by Borcherds [1].)

If we choose the point $(x, y) = (-0.6442, -0.9195)$, we find that $R = -1.31996$ and $d = 0.2324$. This requires a slight adjustment for drawing the circles, as we can't use a negative radius:

```
c1:=Circle[(0,0),abs(R)]
```

Plot the circle c2 as before, and choose any point on the outer circle for the start; one choice is shown in figure 7(b). Convex bicentric polygons with n sides exist for all n, and bicentric n-pointed stars will exist for $n = 5$, as we have seen, and for all $n \geq 7$.

We finish by noting a remarkable recent result [2], which gives a complete characterization for a bicentric polygon to exist. Given R, r and d, define the function

$$b(t) = \left[1 - \left(\frac{r - d\cos t}{R}\right)^2\right]^{-1/2}$$

and the value

$$\alpha = 2\tan^{-1}\left(\frac{1}{r}\sqrt{R^2 - (r-d)^2}\right).$$

Then define

$$B(x) = \int_0^x b(t)\,dt.$$

Then an n-sided bicentric polygon between the circles $x^2 + y^2 = R^2$ and $(x - d)^2 + y^2 = r^2$ will exist if and only if

$$\frac{B(2\pi)}{B(\alpha)} = n.$$

For a proof, see the reference cited. It is not yet known whether this result will lead to simpler, or new, relationships between R, r and d for n-sided bicentric polygons.

5 Conclusions

We have shown that the difficult algebra and geometry of bicentric polygons is easy to manage with dynamic geometry software, especially if the number of edges is not too large. But there is no reason why formulas corresponding to those of Euler and Fuss for larger numbers of edges can't be also easily obtained. Although many such formulas have been developed, it is not known which are the simplest, or even if there exists a simper version of some of the higher order examples. And of course for large n there are polygons and stars of various sorts to be drawn. The use of modern software means that the extremely complicated algebraic manipulations can be automated by the system, and the exploration can concentrate on experimenting with the geometry. There is no reason why (some of) this material would not make an excellent investigative project for a secondary student, or a school mathematics club.

Acknowledgements

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